# ON THE CONSTRUCTION OF PERIODIC SOLUTIONS OF A NONAUTONOMOUS QUASILINEAR SYSTEM WITH TWO DEGREES OF FREEDOM

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*PMM Vol.27, No.2, 1963, pp. 365-369*

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*(Received June 14, 1962)*

The following nonautonomous quasilinear system with two degrees of freedom is considered

$$\begin{aligned} \ddot{x} + ax + by &= f(t) + \mu F(t, x, \dot{x}, y, \dot{y}, \mu) \\ \ddot{y} + cx + dy &= \varphi(t) + \mu\Phi(t, x, \dot{x}, y, \dot{y}, \mu) \end{aligned} \tag{0.1}$$

It is assumed that $f$ and $\varphi$ are continuous periodic functions of period $2\pi$; the functions $F$ and $\Phi$ are analytic with respect to the variables $x$, $\dot{x}$, $y$, $\dot{y}$ and $\mu$ and are continuous functions of $t$ having the same period $2\pi$. The quantity $\mu$ is a small parameter. The coefficients $a$, $b$, $c$ and $d$ are constants. We shall consider the case of resonance [1, p. 107], when the fundamental equation

$$(D^2 + a)(D^2 + d) - bc = 0$$

has two roots equal to $\pm ik$, where $k$ is an integer, and two other roots equal to $\pm i\omega$, where $\omega$ is not an integer.

The general solution of the generating system ($\mu = 0$) has the form

$$\begin{aligned} x_0^*(t) &= A_0 \cos kt + \frac{B_0}{k}\sin kt + E_0 \cos\omega t + \frac{D_0}{\omega}\sin\omega t + f^*(t) \\ y_0^*(t) &= p_1\left(A_0 \cos kt + \frac{B_0}{k}\sin kt\right) + p_2\left(E_0\cos\omega t + \frac{D_0}{\omega}\sin\omega t\right) + \varphi^*(t) \end{aligned} \tag{0.2}$$

where $f^*(t)$ and $\varphi^*(t)$ are particular solutions of the generating system, $A_0$, $B_0$, $E_0$ and $D_0$ are arbitrary constants

$$p_1 = \frac{c}{k^2 - d} = \frac{k^2 - a}{b}, \qquad p_2 = \frac{c}{\omega^2 - d} = \frac{\omega^2 - a}{b}$$

We shall separate from (0.2) a family of periodic solutions of period $2\pi$

$$x_0(t) = A_0 \cos kt + \frac{B_0}{k} \sin kt + f^\circ(t)$$
$$y_0(t) = p_1\left(A_0 \cos kt + \frac{B_0}{k} \sin kt\right) + \varphi^\circ(t) \tag{0.3}$$

Here $f^\circ(t)$ and $\varphi^\circ(t)$ represent a particular solution of period $2\pi$ of the system (0.1) for $\mu = 0$. According to [1, p.109], the necessary and sufficient condition for the system (0.1) for $\mu = 0$ to have, in the case of the resonance, the periodic solutions (0.3), is that the functions $f(t)$ and $\varphi(t)$ should satisfy the two conditions

$$\int_0^{2\pi} \left[f(\tau) + \frac{b}{k^2 - d}\varphi(\tau)\right] \cos k\tau d\tau = 0, \qquad \int_0^{2\pi} \left[f(\tau) + \frac{b}{k^2 - d}\varphi(\tau)\right] \sin k\tau d\tau = 0$$

The problem consists in the search of periodic solutions of period $2\pi$ of the system (0.1), which corresponds to the generating solution (0.3) when $\mu = 0$. In [2] it was erroneously indicated that the solution (0.1) has a form analogous to the form of the solution (0.3) of the generating system. Actually it is not so. We shall determine the existence conditions of such periodic solutions of (0.1) and shall show how they can be determined.

**1.** In accordance with Poincaré's method, initial conditions for the system (0.1) are taken in the form

$$x(0) = x_0(0) + b_1, \qquad \dot{x}(0) = \dot{x}_0(0) + b_2$$
$$y(0) = y_0(0) + b_3, \qquad \dot{y}(0) = \dot{y}_0(0) + b_4 \tag{1.1}$$

where the $b_i$'s are some quantities, vanishing for $\mu = 0$. Then the solution of the system (0.1) depends upon the parameters $b_1$, $b_2$, $b_3$, $b_4$ and can be expanded in series of integer powers of these parameters

$$x(t) = x_0(t) + \sum_{i=1}^{4} P_{1i}(t)\, b_i + \mu\,[\ldots] = 0$$
$$y(t) = y_0(t) + \sum_{i=1}^{4} P_{2i}(t)\, b_i + \mu\,[\ldots] = 0 \tag{1.2}$$

The functions $P_{1i}(t)$ and $P_{2i}(t)$ are found by substituting the series of (1.2) in equation (0.1) and equating the coefficients of the terms of like powers in $b_i$ and $\mu$. Let us introduce the notations

$$\frac{1}{p_1-p_2}(b_3-p_2b_1)=\beta_1, \qquad \frac{1}{p_1-p_2}(p_1b_1-b_3)=\beta_3$$
$$\frac{1}{p_1-p_2}(b_4-p_2b_2)=\beta_2, \qquad \frac{1}{p_1-p_2}(p_1b_2-b_4)=\beta_4 \tag{1.3}$$

As a result we shall have the solution of (1.2) in the form

$$x(t)=x_0(t)+\beta_1\cos kt+\frac{\beta_2}{k}\sin kt+\beta_3\cos\omega t+\frac{\beta_4}{\omega}\sin\omega t+\mu[\ldots]$$
$$y(t)=y_0(t)+p_1\left(\beta_1\cos kt+\frac{\beta_2}{k}\sin kt\right)+p_2\left(\beta_3\cos\omega t+\frac{\beta_4}{\omega}\sin\omega t\right)+\mu[\ldots] \tag{1.4}$$

The initial conditions (1.1) become

$$x(0)=x_0(0)+\beta_1+\beta_3, \qquad y(0)=y_0(0)+p_1\beta_1+p_2\beta_3$$
$$\dot x(0)=\dot x_0(0)+\beta_2+\beta_4, \qquad \dot y(0)=\dot y_0(0)+p_1\beta_2+p_2\beta_4 \tag{1.5}$$

As shown in [1, p.119] two of the quantities $\beta_i$ (their number is equal to the number of arbitrary constants entering the generating solution) are analytic functions of the two others and of the parameter $\mu$, and become zero for $\mu = 0$. Here $\beta_1$ and $\beta_2$ are the independent quantities and $\beta_3$ and $\beta_4$ the analytic functions of $\beta_1$, $\beta_2$ and $\mu$. In fact, we shall write the conditions of periodicity of the functions $x(t, \beta_i, \mu)$ and $\dot x(t, \beta_i, \mu)$ in accordance with (1.4) and (1.5)

$$\beta_3(\cos 2\pi\omega-1)+\frac{\beta_4}{\omega}\sin 2\pi\omega+\Theta_1(\beta_1,\beta_2,\beta_3,\beta_4,\mu)=0$$
$$-\beta_3\omega\sin 2\pi\omega+\beta_4(\cos 2\pi\omega-1)+\Theta_2(\beta_1,\beta_2,\beta_3,\beta_4,\mu)=0 \tag{1.6}$$

The functions $\Theta_1$ and $\Theta_2$ are some analytic functions of $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$ and $\mu$. The conditions of periodicity of the functions $y(t, \beta_i, \mu)$ and $\dot y(t, \beta_i, \mu)$ differ from the conditions (1.6) only by a factor $p_2$. Since $2(1 - \cos 2\pi\omega) \neq 0$, the equations (1.6) can be solved with respect to $\beta_3$ and $\beta_4$, and yield the analytic functions

$$\beta_3=\psi_1(\beta_1,\beta_2,\mu), \qquad \beta_4=\psi_2(\beta_1,\beta_2,\mu)$$

which become zero for $\mu = 0$.

Thus, the solution of (1.4) is represented by the series

$$x(t)=x_0(t)+\beta_1\cos kt+\frac{\beta_2}{k}\sin kt+\psi_1\cos\omega t+\frac{\psi_2}{\omega}\sin\omega t+$$
$$+\sum_{n=1}^{\infty}\left[C_n(t)+\frac{\partial C_n(t)}{\partial A_0}\beta_1+\frac{\partial C_n(t)}{\partial B_0}\beta_2+\ldots\right]\mu^n \tag{1.7}$$

$$y(t) = y_0(t) + p_1\left(\beta_1 \cos kt + \frac{\beta_2}{k}\sin kt\right) + p_2\left(\psi_1 \cos \omega t + \frac{\psi_2}{\omega}\sin \omega t\right) + \\ + \sum_{n=1}^{\infty}\left[H_n(t) + \frac{\partial H_n(t)}{\partial A_0}\beta_1 + \frac{\partial H_n(t)}{\partial B_0}\beta_2 + \ldots\right]\mu^n \tag{1.7}$$

The functions $C_n(t)$ and $H_n(t)$ are determined as in [2]

$$C_n(t) = C_n^{(1)}(t) + C_n^{(2)}(t), \qquad H_n(t) = p_1 C_n^{(1)}(t) + p_2 C_n^{(2)}(t)$$

where

$$C_n^{(1)}(1) = \frac{1}{\omega^2 - k^2}\int_0^t \left[\frac{d-k^2}{k} F_n(\tau) - \frac{b}{k}\Phi_n(\tau)\right]\sin k(t-\tau)\, d\tau$$

$$C_n^{(2)}(t) = -\frac{1}{\omega^2 - k^2}\int_0^t \left[\frac{d-\omega^2}{\omega} F_n(\tau) - \frac{b}{\omega}\Phi_n(\tau)\right]\sin \omega(t-\tau)\, d\tau \tag{1.8}$$

Here

$$F_n(t) = \frac{1}{(n-1)!}\left(\frac{d^{n-1}F}{d\mu^{n-1}}\right)_{\beta_i=\mu=0}, \qquad \Phi_n(t) = \frac{1}{(n-1)!}\left(\frac{d^{n-1}\Phi}{d\mu^{n-1}}\right)_{\beta_i=\mu=0}$$

Expanding $F_n(t)$ and $\Phi_n(t)$ in Fourier series and calculating the integrals of (1.8), it is easy to show that $C_n^{(2)}(t)$ has for components a periodic function of period $2\pi$ and the harmonics of $\sin \omega t$ and $\cos \omega t$ with some coefficients. In other words, $C_n^{(2)}(t)$ is not a periodic function of period $2\pi$, since $\omega$ is not an integer.

The solution of (1.7) is represented as

$$x(t) = x_0(t) + x^{(1)}(t) + x^{(2)}(t), \quad y(t) = y_0(t) + p_1 x^{(1)}(t) + p_2 x^{(2)}(t) \tag{1.9}$$

where

$$x^{(1)}(t) = \beta_1 \cos kt + \frac{\beta_2}{k}\sin kt + \sum_{n=1}^{\infty}\left[C_n^{(1)}(t) + \frac{\partial C_n^{(1)}(t)}{\partial A_0}\beta_1 + \frac{\partial C_n^{(1)}(t)}{\partial B_0}\beta_2 + \ldots\right]\mu^n$$

$$x^{(2)}(t) = \psi_1 \cos \omega t + \frac{\psi_2}{\omega}\sin \omega t + \sum_{n=1}^{\infty}\left[C_n^{(2)}(t) + \frac{\partial C_n^{(2)}(t)}{\partial A_0}\beta_1 + \frac{\partial C_n^{(2)}(t)}{\partial B_0}\beta_2 + \ldots\right]\mu^n \tag{1.10}$$

The necessary and sufficient condition for obtaining a periodic solution of period $2\pi$ for (1.9), is that the conditions of periodicity of the functions $x^{(1)}(t)$ and $x^{(2)}(t)$ be satisfied

$$x^{(1)}(2\pi) = \beta_1, \qquad \dot{x}^{(1)}(2\pi) = \beta_2 \tag{1.11}$$

$$x^{(2)}(2\pi) = \psi_1(\beta_1, \beta_2, \mu), \qquad \dot{x}^{(2)}(2\pi) = \psi_2(\beta_1, \beta_2, \mu) \tag{1.12}$$

Thus, the problem has reduced to the construction of the periodic functions $x^{(1)}(t)$ and $x^{(2)}(t)$ of period $2\pi$.

2. The construction of the function $x^{(1)}(t)$ is analogous to the construction of the periodic solution of a quasilinear nonautonomous system with one degree of freedom [3, 4]. Thus, from the conditions of periodicity (1.11) it is possible to determine the amplitudes $A_0$ and $B_0$ of the generating function and the quantities $\beta_1$ and $\beta_2$ in the form of a series of integer or fractional powers of $\mu$, depending upon the multiplicity of the roots of the equations of the basic amplitudes $C_1^{(1)}(2\pi) = 0$ and $\dot{C}_1^{(1)}(2\pi) = 0$.

3. The construction of the function $x^{(2)}(t)$ is made according to (1.10) if the quantities $\psi_1$ and $\psi_2$ are determined. Therefore, by virtue of the analicity of $\psi_1$ and $\psi_2$ with respect to $\beta_1$, $\beta_2$ and $\mu$, and also since a differentiation with respect to $\beta_1$ and $\beta_2$ can be replaced by a differentiation with respect to $A_0$ and $B_0$, we have

$$\psi_j(\beta_1, \beta_2, \mu) = \sum_{n=1}^{\infty} \left[ \Psi_{jn} + \frac{\partial \Psi_{jn}}{\partial A_0}\beta_1 + \frac{\partial \Psi_{jn}}{\partial B_0}\beta_2 + \ldots \right] \mu^n \qquad (j = 1, 2) \tag{3.1}$$

The conditions (1.12) of periodicity of the functions $x^{(2)}(t)$ and $\dot{x}^{(2)}(t)$ are used for the determination of $\psi_{jn}$

$$\Psi_{1n}(\cos 2\pi\omega - 1) + \frac{\Psi_{2n}}{\omega}\sin 2\pi\omega + C_n^{(2)}(2\pi) = 0$$

$$-\omega\Psi_{1n}\sin 2\pi\omega + \Psi_{2n}(\cos 2\pi\omega - 1) + \dot{C}_n^{(2)}(2\pi) = 0$$

Whereupon

$$\Psi_{1n} = \frac{1}{2}\left[ C_n^{(2)}(2\pi) + \frac{\dot{C}_n^{(2)}(2\pi)}{\omega}\cot \pi\omega \right], \qquad \Psi_{2n} = \frac{1}{2}\left[ \dot{C}_n^{(2)}(2\pi) - \omega C_n^{(2)}(2\pi)\cot \pi\omega \right] \tag{3.2}$$

Thus, $\psi_{1n}$ and $\psi_{2n}$ are calculated from the formulas (3.2) once the functions $C_n^{(2)}(t)$ and $\dot{C}_n^{(2)}(t)$ are known. Knowing $\psi_{1n}$ and $\psi_{2n}$, we determine on the basis of (3.1) the quantities $\psi_j(\beta_1, \beta_2, \mu)$ and also the function $x^{(2)}(t)$ from the second formula (1.10).

In order to determine the functions $C_n^{(1)}(t)$ and $C_n^{(2)}(t)$ it is indispensable to know $F_n(t)$ and $\Phi_n(t)$. We shall denote

$$C_n^{*(2)}(t) = C_n^{(2)}(t) + \Psi_{1n}\cos\omega t + \frac{\Psi_{2n}}{\omega}\sin\omega t$$

$$C_n^*(t) = C_n^{(1)}(t) + C_n^{*(2)}(t), \quad H_n^*(t) = p_1 C_n^{(1)}(t) + p_2 C_n^{*(2)}(t)$$

By verification we determine that the functions $C_n^{*(2)}(t)$ are periodic of period $2\pi$. The expressions for the functions $F_n(t)$ and $\Phi_n(t)$ are obtained from the corresponding expressions of [2], replacing $C_n(t)$, $\dot{C}_n(t)$, $H_n(t)$ and $\dot{H}_n(t)$ by $C_n^*(t)$, $\dot{C}_n^*(t)$, $H_n^*(t)$ and $\dot{H}_n^*(t)$.

4. If the quantities $\beta_1$ and $\beta_2$ are determined by series [3] of integer powers of the parameter $\mu$

$$\beta_1 = \sum_{n=1}^{\infty} A_n\mu^n, \qquad \beta_2 = \sum_{n=1}^{\infty} B_n\mu^n$$

then the functions $x^{(1)}(t)$ and $x^{(2)}(t)$, and consequently the solution $x(t)$ and $y(t)$ are series of integer powers of the parameter $\mu$

$$x^{(1)}(t) = \mu x_1^{(1)}(t) + \mu^2 x_2^{(1)}(t) + \ldots, \qquad x^{(2)}(t) = \mu x_1^{(2)}(t) + \mu^2 x_2^{(2)}(t) + \ldots$$

We shall determine the expressions of the first two functions

$$x_1^{(1)}(t) = A_1\cos kt + \frac{B_1}{k}\sin kt + C_1^{(1)}(t)$$

$$x_2^{(1)}(t) = A_2\cos kt + \frac{B_2}{k}\sin kt + C_2^{(1)}(t) + \frac{\partial C_1^{(1)}(t)}{\partial A_0}A_1 + \frac{\partial C_1^{(1)}(t)}{\partial B_0}B_1 \quad \text{etc.}$$

$$x_1^{(2)}(t) = C_1^{*(2)}(t), \; x_2^{(2)}(t) = C_2^{*(2)}(t) + \frac{\partial C_1^{*(2)}(t)}{\partial A_0}A_1 + \frac{\partial C_1^{*(2)}(t)}{\partial B_0}B_1 \quad \text{etc.}$$

If the equations of the basic amplitudes have double roots, then the quantities $\beta_1$ and $\beta_2$ are sought for in the form of series in the powers of $\mu$ and $\mu^{1/2}$. The function $x^{(1)}(t)$ is found in a manner similar to that used in [4] and $x^{(2)}(t)$, as was shown in Section 3.

The method of construction of the periodic solutions can be extended to systems with $n$ degrees of freedom. For instance, in the case of oscillations of same frequencies, the construction of the periodic solutions of period $2\pi$ breaks down into $n$ separate problems of successive determination of periodic functions $x^{(1)}(t), \ldots, x^{(n)}(t)$. Thus the problem of the construction of $x^{(1)}(t)$ is similar to the search of a periodic solution of a system with one degree of freedom, and the others are found by the method of Section 3.

*Example*. The following system of equations is considered

$$\ddot{x} + y = \cos 2t + \mu (1 - x^2)\dot{x}, \qquad \ddot{y} - \frac{1}{4}x + \frac{5}{4}y = -\frac{11}{4}\cos 2t + \mu\dot{y}$$

The fundamental equation has for roots $\pm\ i$ and $\pm\ 1/2\ i$. Subharmonic solutions of the system are sought. We have $p_1 = 1$ and $p_2 = 1/4$ The generating solution depends upon two arbitrary constants

$$x_0 = A_0 \cos t + B_0 \sin t, \qquad y_0 = A_0 \cos t + B_0 \sin t + \cos 2t$$

The equations of the basic amplitudes

$$A_0\left[3 + \frac{1}{4}(A_0^2 + B_0^2)\right] = 0, \qquad B_0\left[3 + \frac{1}{4}(A_0^2 + B_0^2)\right] = 0$$

have the obvious solution $A_0 = B_0 = 0$. Using the construction procedure of periodic solutions described above, we obtain the first approximation

$$x(t) = \left[\frac{16}{9}\left(\frac{\pi}{3} - 1\right)\sin t + \frac{8}{5}\sin 2t\right]\mu$$

$$y(t) = \cos 2t + \left[\frac{16}{9}\left(\frac{\pi}{3} - 1\right)\sin t + \frac{16}{15}\sin 2t\right]\mu$$

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*Translated by* A.V.